Métodos de elementos finitos mixtos para problemas no uniformemente elípticos

Ricardo G. Durán

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
IMAS, UBA-CONICET

IMAL, Santa Fe
September 2, 2016

SUMMARY

- Anisotropic error estimates for mixed methods: review of several arguments.

SUMMARY

- Anisotropic error estimates for mixed methods: review of several arguments.
- Degenerate elliptic problems.

SUMMARY

- Anisotropic error estimates for mixed methods: review of several arguments.
- Degenerate elliptic problems.
- Weighted Poincaré type inequalities.

- Anisotropic error estimates for mixed methods: review of several arguments.
- Degenerate elliptic problems.
- Weighted Poincaré type inequalities.
- Weighted error estimates for the Raviart-Thomas interpolation.

SUMMARY

- Anisotropic error estimates for mixed methods: review of several arguments.
- Degenerate elliptic problems.
- Weighted Poincaré type inequalities.
- Weighted error estimates for the Raviart-Thomas interpolation.
- Application to the Fractional Laplacian.

For $k = 0, 1, 2, \cdots$

$$\mathcal{R}T_k(T) = \mathcal{P}_k^2(T) \oplus (x, y)\mathcal{P}_k(T)$$

and its extension to tetrahedra (Nedelec),

$$\mathcal{R}T_k(T) = \mathcal{P}_k^3(T) \oplus (x, y, z)\mathcal{P}_k(T)$$

RAVIART-THOMAS INTERPOLATION

$$RT_k \;:\; H^1(T)^n \rightarrow \mathcal{RT}_k(T)$$

Face (or edge if $n = 2$) degrees of freedom:

$$\int_{F_i} RT_k \boldsymbol{\sigma} \cdot n_i \, p_k \, ds = \int_{F_i} \boldsymbol{\sigma} \cdot n_i \, p_k \, ds \;\; \forall p_k \in \mathcal{P}_k(F_i)$$

Internal degrees of freedom (for $k \geq 1$)

$$\int_T RT_k \boldsymbol{\sigma} \cdot \mathbf{p}_{k-1} \, dx = \int_T \boldsymbol{\sigma} \cdot \mathbf{p}_{k-1} \, dx \;\; \forall \mathbf{p}_{k-1} \in \mathcal{P}^n_{k-1}(T)$$

FUNDAMENTAL PROPERTY

$$\int_T \mathrm{div}\,(\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma})\,q = 0 \qquad \forall q \in \mathcal{P}_k(T)$$

i.e.,

$$\mathrm{div}\,RT_k\boldsymbol{\sigma} = P_k \mathrm{div}\;\boldsymbol{\sigma}$$

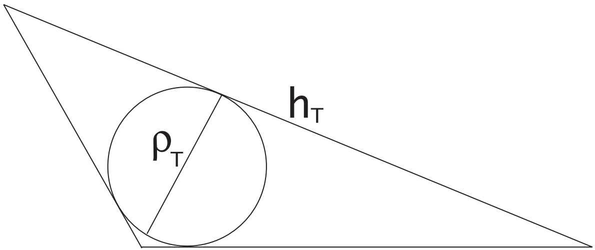
where

$$P_k \;:\; L^2(T) \to \mathcal{P}_k$$

is the L^2-orthogonal projection.

REGULARITY ASSUMPTION

Raviart-Thomas (1975), Nedelec (1980)



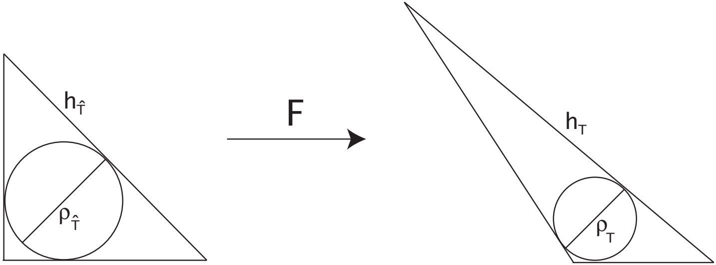
REGULARITY ASSUMPTION

h_T exterior diameter, ρ_T interior diameter

$$\frac{h_T}{\rho_T} \leq \gamma$$

The constant in the error estimates depends on the regularity parameter γ

STANDARD ARGUMENTS



$\widehat{T}$ reference element $\quad F : \widehat{T} \to T \quad$ affine transformation

The Piola transform preserves the degrees of freedom!

$$\boldsymbol{\sigma}(x) = \frac{1}{|\det DF(\hat{x})|} DF(\hat{x})\hat{\boldsymbol{\sigma}}(\hat{x})$$

where $x = F(\hat{x})$.

$$\widehat{RT_k \boldsymbol{\sigma}} = \widehat{RT_k} \hat{\boldsymbol{\sigma}}$$

ERROR ESTIMATES

Polynomial approximation $+$ Piola transform $\Rightarrow$

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \leq C(\gamma)\, h_T^m \|D^m\boldsymbol{\sigma}\|_{L^2(T)}$$

$$1 \leq m \leq k+1$$

In the case of standard Lagrange interpolation it is known that the regularity condition can be relaxed
Babuska-Aziz, Jamet, Krizek, Al Shenk, Dobrowolski, Apel, Nicaise, Formaggia, Perotto, Acosta, Lombardi, D., etc..

Is it possible to relax the regularity condition for RT interpolation?

YES!

We developed several arguments to obtain estimates in 2 and 3 dimensions.

CASE $k = 0$, $n = 2$

We work in a family of reference elements and use the Piola transform associated with

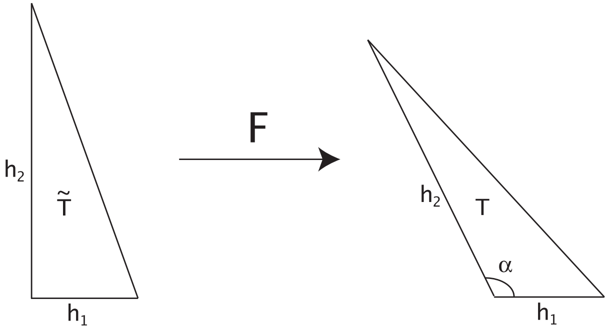
$$F : \widetilde{T} \to T$$

$$F(\tilde{x}) = M\tilde{x} + b$$

with

$$\|M\|, \|M^{-1}\| \leq C$$

CASE $k = 0$, $n = 2$



REMARK:

In this way we obtain from the reference family the family of all elements with maximum angle α satisfying $\alpha < \psi(C) < \pi$

MAXIMUM ANGLE CONDITION

From the definition of RT_0 on the reference element

$$\int_{\ell_i} (\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}) \cdot \nu_i = 0 \qquad \forall \ell_i \quad \text{edge of} \quad \widetilde{T}$$

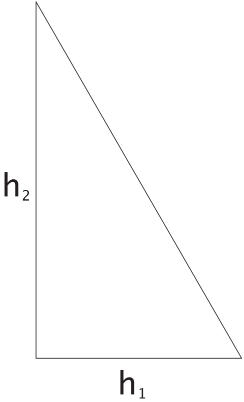
Then, if ℓ_i is the edge with normal $\mathbf{e_i}$,

$$\int_{\ell_1} (\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_2 = 0$$

and

$$\int_{\ell_2} (\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_1 = 0$$

We use the Poincaré type inequality on



CASE $k = 0$, $n = 2$

$$\int_{\ell} v = 0 \Longrightarrow$$

$$\|v\|_{L^2(\widetilde{T})} \leq C \left\{ h_1 \left\| \frac{\partial v}{\partial x} \right\|_{L^2(\widetilde{T})} + h_2 \left\| \frac{\partial v}{\partial y} \right\|_{L^2(\widetilde{T})} \right\}$$

Taking

$$v = (\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_i$$

we obtain

$$\|(\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_i\|_{L^2(\widetilde{T})} \leq C \left\{ h_1 \left\| \frac{\partial(\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_i}{\partial x} \right\|_{L^2(\widetilde{T})} + h_2 \left\| \frac{\partial(\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma})_i}{\partial y} \right\|_{L^2(\widetilde{T})} \right\}$$

We have to eliminate the dependence on $RT_0\boldsymbol{\sigma}$ from the right hand side

But,

$$\frac{\partial(RT_0\boldsymbol{\sigma})_1}{\partial x} = \frac{\partial(RT_0\boldsymbol{\sigma})_2}{\partial y} = \frac{\operatorname{div} RT_0\boldsymbol{\sigma}}{2}$$

and

$$\frac{\partial(RT_0\boldsymbol{\sigma})_1}{\partial y} = \frac{\partial(RT_0\boldsymbol{\sigma})_2}{\partial x} = 0$$

but, from the commutative diagram property:

$$\operatorname{div} RT_0\boldsymbol{\sigma} = P_0 \operatorname{div} \boldsymbol{\sigma}$$

and therefore,

$$\|\operatorname{div} RT_0\boldsymbol{\sigma}\|_{L^2(\widetilde{T})} \leq \|\operatorname{div} \boldsymbol{\sigma}\|_{L^2(\widetilde{T})}$$

Then,

$$\|\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}\|_{L^2(\widetilde{T})}$$

$$\leq C\left\{h_1\left\|\frac{\partial\boldsymbol{\sigma}}{\partial x}\right\|_{L^2(\widetilde{T})} + h_2\left\|\frac{\partial\boldsymbol{\sigma}}{\partial y}\right\|_{L^2(\widetilde{T})} + (h_1 + h_2)\|\operatorname{div} \boldsymbol{\sigma}\|_{L^2(\widetilde{T})}\right\}$$

Therefore, using the Piola transform we obtain,

$$\|\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}\|_{L^2(T)} \leq \frac{C}{\sin\alpha} h_T \|D\boldsymbol{\sigma}\|_{L^2(T)}$$

for a general triangle T with maximum angle α.

Applying similar arguments than for RT_0, i. e.,

A generalized Poincaré inequality

For example, for $k = 1$

$$\int_\ell v\, p_1 = 0 \quad \forall p_1 \in \mathcal{P}_1(\ell) \quad , \quad \int_{\widetilde{T}} v = 0 \quad \Longrightarrow$$

$$\|v\|_{L^2(\widetilde{T})} \le C \sum_{i,j=1}^{2} h_i h_j \left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_{L^2(\widetilde{T})}$$

We obtain

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le Ch_T^{k+1}\|D^{k+1}(\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma})\|_{L^2(T)}$$

under the MAXIMUM ANGLE CONDITION

We obtain

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \leq Ch_T^{k+1}\|D^{k+1}(\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma})\|_{L^2(T)}$$

under the MAXIMUM ANGLE CONDITION

How do we bound $\|D^{k+1}RT_k\boldsymbol{\sigma}\|_{L^2(T)}$?

We use

$$D^{k+1} RT_k \boldsymbol{\sigma} = D^k \text{div}\, RT_k \boldsymbol{\sigma}$$

But,

$$\text{div}\ RT_k \boldsymbol{\sigma} = P_k \text{div}\ \boldsymbol{\sigma}$$

and then,

$$\|D^{k+1} RT_k \boldsymbol{\sigma}\|_{L^2(T)} \leq C \|D^k P_k \text{div}\, \boldsymbol{\sigma}\|_{L^2(T)}$$

But, we can prove

$$\|D^k P_k f\|_{L^2(T)} \leq C(\alpha)\|D^k f\|_{L^2(T)}$$

where α is the maximum angle of T

REMARK: An analogous estimate can be obtained by using inverse inequalities, but in this way the constant would depend on the minimum angle!

Summing up we obtain

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le C h_T^{k+1} \|D^{k+1}\boldsymbol{\sigma}\|_{L^2(T)}$$

under the MAXIMUM ANGLE CONDITION

Summing up we obtain

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le Ch_T^{k+1}\|D^{k+1}\boldsymbol{\sigma}\|_{L^2(T)}$$

under the MAXIMUM ANGLE CONDITION

This analysis is simple but has some important drawbacks!

It does not apply to obtain

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le Ch_T^{m}\|D^{m}\boldsymbol{\sigma}\|_{L^2(T)}, \quad 1 \le m \le k+1$$

HIGHER ORDER RT ELEMENTS

In particular $m = 1$

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \leq Ch_T\|D\boldsymbol{\sigma}\|_{L^2(T)}$$

$$\Longrightarrow \quad INF - SUP$$

IMPORTANT IN ERROR ANALYSIS!

In particular $m = 1$

$$\|\boldsymbol{\sigma} - RT_k\boldsymbol{\sigma}\|_{L^2(T)} \leq Ch_T\|D\boldsymbol{\sigma}\|_{L^2(T)}$$

$$\Longrightarrow \quad INF - SUP$$

IMPORTANT IN ERROR ANALYSIS!

Moreover,

The extension of the arguments to the 3d case does not give a complete result: It only applies to a restricted class of elements

Two generalizations of the MAXIMUM ANGLE CONDITION:

- REGULAR VERTEX PROPERTY
  A family of tetrahedra satisfies the RVP if for some vertex, the three edges containing that vertex remain “Uniformly linearly independent”.

- MAXIMUM ANGLE CONDITION
  A family of tetrahedra satisfies the MAC if the angles between edges and between faces remain uniformly bounded away from π.

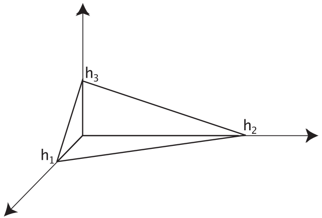
THE 3D CASE

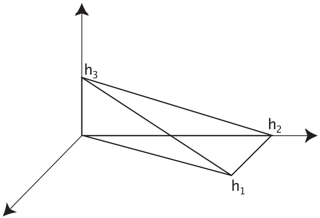
In 2D $\qquad$ RVP $\iff$ MAC

In 3D $\qquad$ RVP $\Longrightarrow$ MAC

BUT NOT CONVERSELY!

Now we have to work with two families of reference elements





Using the Piola transform associated with

$$F : \widetilde{T} \to T$$

$$F(\tilde{x}) = M\tilde{x} + b$$

with

$$\|M\|, \|M^{-1}\| \leq C$$

we obtain RVP from the left family and MAC from the union of both families.

THE 3D CASE

Using the Piola transform associated with

$$F : \widetilde{T} \to T$$

$$F(\tilde{x}) = M\tilde{x} + b$$

with

$$\|M\|, \|M^{-1}\| \leq C$$

we obtain RVP from the left family and MAC from the union of

both families.

A straightforward generalization of the argument given in 2D proves the error estimate under the RVP property!

Using the Piola transform associated with

$$F : \widetilde{T} \to T$$

$$F(\tilde{x}) = M\tilde{x} + b$$

with

$$\|M\|, \|M^{-1}\| \leq C$$

we obtain RVP from the left family and MAC from the union of

both families.

A straightforward generalization of the argument given in 2D proves the error estimate under the RVP property!

QUESTIONS

1. It is possible to obtain error estimates under MAC in 3D ?
2. It is possible to obtain error estimates for less regular functions?

QUESTIONS

1. It is possible to obtain error estimates under MAC in 3D ?
2. It is possible to obtain error estimates for less regular functions?

Yes!

But we need a different argument.

Idea: Reduction to a finite dimensional problem

Case $k = 0$ in 3D

We use the face average interpolant

$$\Pi : H^1(T)^3 \to \mathcal{P}_1(T)^3$$

$$\int_S \Pi\boldsymbol{\sigma} = \int_S \boldsymbol{\sigma}$$

Properties of Π:

- $\|D\Pi\boldsymbol{\sigma}\|_{L^2(T)} \leq \|D\boldsymbol{\sigma}\|_{L^2(T)}$
- $\|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{L^2(T)} \leq Ch_T\|D\boldsymbol{\sigma}\|_{L^2(T)}$ C independent of the shape!
- $RT_0\boldsymbol{\sigma} = RT_0\Pi\boldsymbol{\sigma}$

Case $n = 3$, $k = 0$

If

$$\|\boldsymbol{\tau} - RT_0\boldsymbol{\tau}\|_{L^2(T)} \le C_1 h_T \|D\boldsymbol{\tau}\|_{L^2(T)} \qquad \forall \boldsymbol{\tau} \in \mathcal{P}_1(T)^3$$

Then,

$$\|\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}\|_{L^2(T)} \le (C + C_1) h_T \|D\boldsymbol{\sigma}\|_{L^2(T)} \quad \forall \boldsymbol{\sigma} \in H^1(T)^3$$

with a constant C independent of T!

Proof:

$$\|\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}\|_{L^2(T)} \leq \|\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}\|_{L^2(T)} + \|\Pi\boldsymbol{\sigma} - RT_0\Pi\boldsymbol{\sigma}\|_{L^2(T)}$$

$$\leq Ch_T\|D\boldsymbol{\sigma}\|_{L^2(T)} + C_1 h_T\|D\Pi\boldsymbol{\sigma}\|_{L^2(T)}$$

$$\leq (C + C_1)h_T\|D\boldsymbol{\sigma}\|_{L^2(T)}$$

In this way we obtain

$$\|\boldsymbol{\sigma} - RT_0\boldsymbol{\sigma}\|_{L^2(T)} \leq C(\alpha)\, h_T\|D\boldsymbol{\sigma}\|_{L^2(T)}$$

where α is the maximum angle of T.

Recall the original proof (Raviart-Thomas):

$$\|RT_k\boldsymbol{\sigma}\|_{L^2(\widehat{T})} \leq C\|\boldsymbol{\sigma}\|_{H^1(\widehat{T})}.$$

Complete H^1-norm appears on the right hand side.

$$\Longrightarrow \qquad C = C(\gamma)$$

where γ is the mesh regularity constant.

Idea (in 2D for simplicity):

To obtain sharper estimates on $\widehat{T}$!

Consider the first components

σ_1 and $RT_{k,1}\boldsymbol{\sigma}$

Ideally, we would like

$$\|RT_{k,1}\boldsymbol{\sigma}\|_{L^2(\widehat{T})} \leq C\|\sigma_1\|_{H^1(\widehat{T})}.$$

But it is false:

For example, on the reference triangle:

$$\boldsymbol{\sigma} = (0, y^2) \quad \Longrightarrow \quad RT_0\boldsymbol{\sigma} = \frac{1}{3}(x, y)$$

Which are the essential degrees of freedom defining $RT_{k,1}\boldsymbol{\sigma}$?

To answer this question one can try to “kill” degrees of freedom by modifying $\boldsymbol{\sigma}$ without changing $RT_{k,1}\boldsymbol{\sigma}$.

Which are the essential degrees of freedom defining $RT_{k,1}\boldsymbol{\sigma}$?

To answer this question one can try to "kill" degrees of freedom by modifying $\boldsymbol{\sigma}$ without changing $RT_{k,1}\boldsymbol{\sigma}$.

Key observation:

$$\boldsymbol{\tau} = (0, g(x)) \Longrightarrow RT_{k,1}\boldsymbol{\tau} = 0$$

Then,

$$\boldsymbol{\tau} = (\sigma_1(x,y), \sigma_2(x,y) - u_2(x,0)) \Longrightarrow RT_{k,1}\boldsymbol{\tau} = RT_{k,1}\boldsymbol{\sigma}$$

But,

$$\boldsymbol{\tau} \cdot n = 0 \quad \text{on the edge} \quad \ell_2 \quad \text{contained in} \quad \{y = 0\}$$

Then, the degrees of freedom defining RT_0 associated with that edge vanish!

For $k = 0$ this gives

$$\|RT_{k,1}\boldsymbol{\sigma}\|_{L^2(\widehat{T})} \leq C\{\|\sigma_1\|_{H^1(\widehat{T})} + \|\operatorname{div} \boldsymbol{\sigma}\|_{L^2(\widehat{T})}\}$$

For $k > 0$ we can “kill" internal degrees of freedom:

$$\boldsymbol{\tau} = (\sigma_1(x,y), \sigma_2(x,y) - \sigma_2(x,0) - y q_{k-1}(x,y))$$

with

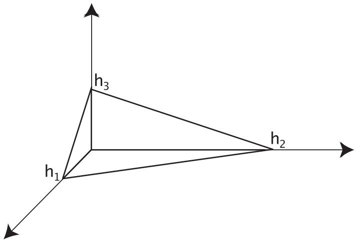
$$q_{k-1} \in \mathcal{P}_{k-1}$$

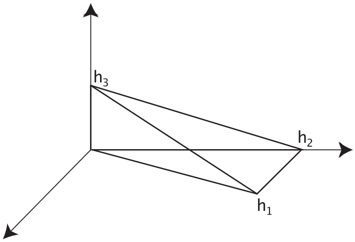
$$\boldsymbol{\tau} \cdot n = 0 \qquad \text{on} \quad \ell_2$$

$$RT_{k,1}\boldsymbol{\tau} = RT_{k,1}\boldsymbol{\sigma}$$

q_{k-1} can be chosen such that internal degrees of freedom corresponding to w_2 vanish!

We apply this argument to the two reference families (we omit details which are rather technical!)





THE 3D CASE

In this way we obtain the first reference family

$$\|RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le C\left\{\|\boldsymbol{\sigma}\|_{L^2(T)} + \sum_{i,j} h_j \left\|\frac{\partial \boldsymbol{\sigma}_i}{\partial x_j}\right\|_{L^2(T)} + h_T\|\operatorname{div}\boldsymbol{\sigma}\|_{L^2(T)}\right\}$$

and analogous estimates under the regular vertex property.

In this way we obtain the first reference family

$$\|RT_k\boldsymbol{\sigma}\|_{L^2(T)} \leq C\left\{\|\boldsymbol{\sigma}\|_{L^2(T)} + \sum_{i,j} h_j \left\|\frac{\partial \boldsymbol{\sigma}_i}{\partial x_j}\right\|_{L^2(T)} + h_T\|\mathrm{div}\,\boldsymbol{\sigma}\|_{L^2(T)}\right\}$$

and analogous estimates under the regular vertex property.

Remark: These error estimates are of anisotropic type!

In this way we obtain the first reference family

$$\|RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le C\left\{\|\boldsymbol{\sigma}\|_{L^2(T)} + \sum_{i,j} h_j \left\|\frac{\partial \boldsymbol{\sigma}_i}{\partial x_j}\right\|_{L^2(T)} + h_T\|\mathrm{div}\,\boldsymbol{\sigma}\|_{L^2(T)}\right\}$$

and analogous estimates under the regular vertex property.

Remark: These error estimates are of anisotropic type!
For maximum angle condition we obtain

$$\|RT_k\boldsymbol{\sigma}\|_{L^2(T)} \le C\left\{\|\boldsymbol{\sigma}\|_{L^2(T)} + h_T\|D\boldsymbol{\sigma}\|_{L^2(T)}\right\}$$

We consider problems of the form

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -A\nabla u \cdot n = f & \text{on } \Gamma_N \end{cases} \tag{1}$$

$$\lambda\omega(x)|\xi|^2 \leq \xi^T \cdot A(x)\xi \leq \Lambda\omega(x)|\xi|^2$$

DEGENERATE ELLIPTIC PROBLEMS

We consider problems of the form

$$\begin{cases} -\text{div}\,(A(x)\nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -A\nabla u \cdot n = f & \text{on } \Gamma_N \end{cases} \tag{1}$$

$$\lambda\omega(x)|\xi|^2 \leq \xi^T \cdot A(x)\xi \leq \Lambda\omega(x)|\xi|^2$$

where ω is a non-negative function that can vanish or become infinity in subsets of $\overline{\Omega}$ with vanishing n-dimensional measure.

DEGENERATE ELLIPTIC PROBLEMS

We consider problems of the form

$$\begin{cases} -\text{div}\,(A(x)\nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -A\nabla u \cdot n = f & \text{on } \Gamma_N \end{cases} \qquad (1)$$

$$\lambda\omega(x)|\xi|^2 \leq \xi^T \cdot A(x)\xi \leq \Lambda\omega(x)|\xi|^2$$

where ω is a non-negative function that can vanish or become infinity in subsets of $\overline{\Omega}$ with vanishing n-dimensional measure.

Typical examples: Examples: $\omega(x) = |x|^\alpha$ or $\omega(x) = dist(x, \Gamma)^\alpha$ with $\Gamma \subset \partial\Omega$.

For simplicity we will consider

$$-\text{div}\,(\omega\nabla u) = g$$

WEIGHTED POINCARÉ INEQUALITIES

This kind of problems were first studied by Fabes, Kenig and Serapioni (1982).

A fundamental tool in their analysis is the Poincaré inequality in weighted norms, namely,

$$\|f - f_\Omega\|_{L^p_\omega(\Omega)} \leq C\|\nabla f\|_{L^p_\omega(\Omega)}$$

WEIGHTED POINCARÉ INEQUALITIES

This kind of problems were first studied by Fabes, Kenig and Serapioni (1982).

A fundamental tool in their analysis is the Poincaré inequality in weighted norms, namely,

$$\|f - f_\Omega\|_{L^p_\omega(\Omega)} \leq C\|\nabla f\|_{L^p_\omega(\Omega)}$$

For our error analysis we need the stronger "Improved Poincaré inequality":

$$\|f - f_\Omega\|_{L^p_\omega(\Omega)} \leq C\|d\nabla f\|_{L^p_\omega(\Omega)}$$

where d is the distance to the boundary.

FKS proved, for Q a cube,

$$\|f - f_Q\|_{L^p_\omega(Q)} \leq C\ell(Q)\|\nabla f\|_{L^p_\omega(Q)}$$

for two classes of weights ω

WEIGHTED POINCARÉ INEQUALITIES

FKS proved, for Q a cube,

$$\|f - f_Q\|_{L^p_\omega(Q)} \leq C\ell(Q)\|\nabla f\|_{L^p_\omega(Q)}$$

for two classes of weights ω

- $\omega \in A_p$

FKS proved, for Q a cube,

$$\|f - f_Q\|_{L^p_\omega(Q)} \leq C\ell(Q)\|\nabla f\|_{L^p_\omega(Q)}$$

for two classes of weights ω

- $\omega \in A_p$
- $\omega = (JF)^{1-p/n}, \quad (1 < p < n)$

  where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a quasi-conformal mapping.

  An interesting example that they give is $\omega = |x|^\alpha$, $\alpha > 0$.

  Actually they proved the result for $n \geq 3$ and $p = 2$, but their argument can be extended straightforward.

IDEA OF THE PROOF FOR $\omega = JF^{1-p/n}$

Using a change of variables, Hölder and that f is quasi-conformal:

$$\int_Q |\varphi(x) - c_Q|^p \omega(x)\, dx \le C\ell(Q)^p \left(\int_{F(Q)} |(\varphi \circ F^{-1})(y) - c_Q|^{p*}\, dy \right)^{\frac{p}{p^*}}$$

IDEA OF THE PROOF FOR $\omega = JF^{1-p/n}$

Using a change of variables, Hölder and that f is quasi-conformal:

$$\int_Q |\varphi(x) - c_Q|^p \omega(x)\, dx \le C\ell(Q)^p \left(\int_{F(Q)} |(\varphi \circ F^{-1})(y) - c_Q|^{p*}\, dy \right)^{\frac{p}{p^*}}$$

$$\int_{F(Q)} |\nabla(\varphi \circ F^{-1})(y)|^p\, dy \le C \int_Q |\nabla \varphi(x)|^p \omega(x)\, dx$$

IDEA OF THE PROOF FOR $\omega = JF^{1-p/n}$

Using a change of variables, Hölder and that f is quasi-conformal:

$$\int_Q |\varphi(x) - c_Q|^p \omega(x)\, dx \leq C\ell(Q)^p \left(\int_{F(Q)} |(\varphi \circ F^{-1})(y) - c_Q|^{p*}\, dy \right)^{\frac{p}{p*}}$$

$$\int_{F(Q)} |\nabla(\varphi \circ F^{-1})(y)|^p\, dy \leq C \int_Q |\nabla\varphi(x)|^p \omega(x)\, dx$$

and therefore, it is enough to prove

$$\left(\int_{F(Q)} |(\varphi \circ F^{-1})(y) - c_Q|^{p*}\, dy \right)^{\frac{1}{p*}} \leq C \left(\int_{F(Q)} |\nabla(\varphi \circ F^{-1})(y)|^p \right)^{\frac{1}{p}}$$

But this is the un-weighted Sobolev-Poincaré in $F(Q)$ which is a John domain.

A REPRESENTATION FORMULA

Suppose that Ω is star-shaped with respect to a ball.

Given a function f we denote with $\bar{f}$ an appropriate weighted average.

A REPRESENTATION FORMULA

Suppose that Ω is star-shaped with respect to a ball.

Given a function f we denote with $\bar{f}$ an appropriate weighted average.

$$f(y) - \bar{f} = - \int_{\Omega} G(x, y) \cdot \nabla f(x) \, dx$$

A REPRESENTATION FORMULA

Suppose that Ω is star-shaped with respect to a ball.

Given a function f we denote with $\bar{f}$ an appropriate weighted average.

$$f(y) - \bar{f} = -\int_\Omega G(x,y) \cdot \nabla f(x)\, dx$$

$$G(x,y) = \int_0^1 \frac{(x-y)}{t} \varphi\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n}$$

It is easy to see that

$$|G(x,y)| \leq \frac{C}{|x-y|^{n-1}}$$

and therefore,

$$|f(y) - \bar{f}| \leq C \int_\Omega \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dx$$

POINCARÉ INEQUALITY

It is easy to see that

$$|G(x,y)| \leq \frac{C}{|x-y|^{n-1}}$$

and therefore,

$$|f(y) - \bar{f}| \leq C \int_\Omega \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dx$$

and the Poincaré inequality

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C\|\nabla f\|_{L^p(\Omega)}$$

follows easily.

It is easy to see that

$$|G(x,y)| \leq \frac{C}{|x-y|^{n-1}}$$

and therefore,

$$|f(y) - \bar{f}| \leq C \int_\Omega \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dx$$

and the Poincaré inequality

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}$$

follows easily.

The weighted case can be proved using results for fractional integrals (this is what FKS did for A_p weights).

But we can do better with a little more effort:

But we can do better with a little more effort:

Using the same representation formula we can prove the Improved Poincaré inequality.

Moreover, the argument can be applied to the weighted case for Muckenhoupt weights.

But we can do better with a little more effort:

Using the same representation formula we can prove the Improved Poincaré inequality.

Moreover, the argument can be applied to the weighted case for Muckenhoupt weights.

We have to use the well known estimate:

$$\int_{|x-y|\leq\varepsilon} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \lesssim \varepsilon Mf(x)$$

IMPROVED POINCARÉ INEQUALITY

Going back to

$$|f(y) - \bar{f}| \lesssim \int_\Omega \frac{|\nabla f(x)|}{|x - y|^{n-1}}\, dx$$

Going back to

$$|f(y) - \bar{f}| \lesssim \int_\Omega \frac{|\nabla f(x)|}{|x - y|^{n-1}}\, dx$$

The key observation is that $G(x, y)$ vanishes for $|x - y| > cd(x)$ (This argument was introduced in Drelichman-D.).

Going back to

$$|f(y) - \bar{f}| \lesssim \int_\Omega \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dx$$

The key observation is that $G(x,y)$ vanishes for $|x-y| > cd(x)$ (This argument was introduced in Drelichman-D.).

Then

$$|f(y) - \bar{f}| \lesssim \int_{|x-y| \lesssim d(x)} \frac{|\nabla f(x)|}{|x-y|^{n-1}}\, dx$$

We use duality,

$$\int_\Omega |f(y) - \bar{f}| g(y)\, dy \lesssim \int_\Omega \int_{|x-y| \lesssim d(x)} \frac{g(y)}{|x-y|^{n-1}}\, dy\, |\nabla f(x)| dx$$

$$\lesssim \int_\Omega d(x)\, Mg(x)\, |\nabla f(x)| dx \leq \|g\|_{L^{p'}(\Omega)} \|d \nabla f\|_{L^p(\Omega)}$$

and then

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C \|d \nabla f\|_{L^p(\Omega)}$$

GENERALIZATION TO JOHN DOMAINS

The representation formula can be generalized replacing segments by appropriate curves.

John domains are those satisfying a “Twisted cone condition”.

For all $y \in \Omega$ there exists a rectifiable curve joining y with a fixed $x_0 \in \Omega$ (we take it $= 0$ to simplify notation), given by a parametrization $\gamma(t, y)$ such that

$$\gamma(0, y) = y \qquad , \qquad \gamma(1, y) = 0$$

and there exist $K, \delta > 0$ such that

$$|\dot{\gamma}(t, y)| \leq K \quad , \quad d(\gamma(t, y)) \geq \delta t$$

With similar arguments we obtain

$$f(y) - \bar{f} = -\int_\Omega G(x,y) \cdot \nabla f(x)\, dx$$

With similar arguments we obtain

$$f(y) - \bar{f} = -\int_\Omega G(x,y)\cdot\nabla f(x)\,dx$$

where now

$$G(x,y) = \int_0^1 \left\{\dot{\gamma}(t,y) + \frac{x-\gamma(t,y)}{t}\right\}\varphi\left(\frac{x-\gamma(t,y)}{t}\right)\frac{dt}{t^n}$$

which satisfies the same properties as in the case of star-shaped domains.

In many applications it is useful to have estimates for less regular functions.

In many applications it is useful to have estimates for less regular functions.

We can generalize the argument to prove improved Poincaré inequalities in fractional Sobolev spaces (Drelichman-D.):

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C|d^s D^s f|_p$$

where we are using the notation

$$|d^s D^s f|_p^p = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} d^s(x) d^s(y)\, dxdy$$

In many applications it is useful to have estimates for less regular functions.

We can generalize the argument to prove improved Poincaré inequalities in fractional Sobolev spaces (Drelichman-D.):

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C|d^s D^s f|_p$$

where we are using the notation

$$|d^s D^s f|_p^p = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} d^s(x) d^s(y)\, dxdy$$

How can we generalize the representation formula?

Idea: regularize f and take "part" of the derivatives to the function used to average:

Define,

$$u(y,t) = (f * \varphi_t)(y)$$

and

$$g(t) = u(\gamma(t,y) + tz, t)$$

Idea: regularize f and take "part" of the derivatives to the function used to average:

Define,

$$u(y,t) = (f * \varphi_t)(y)$$

and

$$g(t) = u(\gamma(t,y) + tz, t)$$

Then,

$$f(y) - (f*\varphi)(z) = u(y,0) - u(z,1) = g(0) - g(1) = -\int_0^1 g'(t)\,dt$$

$$= -\int_0^1 \nabla u(\gamma(t,y) + tz, t) \cdot (\dot{\gamma}(t,z) + z) + u_t(\gamma(t,y) + tz, t)\,dt$$

Multiplying by $\varphi(z)$ and integrating in z we have

$$\begin{aligned} f(y)-c &= \int_{\mathbb{R}^n}(f(y)-(f*\varphi)(z))\varphi(z)\,dz \\ &= -\int_{\mathbb{R}^n}\int_0^1 \nabla u(\gamma(t,y)+tz,t)\cdot(\dot\gamma(t,z)+z)\varphi(z)\,dtdz \\ &\quad -\int_{\mathbb{R}^n}\int_0^1 u_t(\gamma(t,y)+tz,t)\varphi(z)\,dtdz \\ &= I+II \end{aligned}$$

Let us bound for example I (II can be handled analogously).

Changing variables $\gamma(t, y) + tz = x$ and using

$$\frac{\partial u}{\partial x_j}(x, t) = f * (\varphi_t)_{x_j}(x)$$

and

$$(\varphi_t)_{x_j}(x) = \frac{1}{t^{n+1}} \frac{\partial \varphi}{\partial x_j}\left(\frac{x}{t}\right)$$

we have

$$I = \int_0^1 \int_{\mathbb{R}^n} \left(\sum_j \int_{\mathbb{R}^n} f(w) \frac{1}{t^{n+1}} \frac{\partial \varphi}{\partial x_j}\left(\frac{x - w}{t}\right) dw \right) \cdot \psi(x, y, t)\, dx \frac{dt}{t^n}$$

with $\psi(x, y, t)$ bounded

Changing variables $\gamma(t,y)+tz=x$ and using

$$\frac{\partial u}{\partial x_j}(x,t)=f*(\varphi_t)_{x_j}(x)$$

and

$$(\varphi_t)_{x_j}(x)=\frac{1}{t^{n+1}}\frac{\partial\varphi}{\partial x_j}\left(\frac{x}{t}\right)$$

we have

$$I=\int_0^1\int_{\mathbb{R}^n}\left(\sum_j\int_{\mathbb{R}^n}f(w)\frac{1}{t^{n+1}}\frac{\partial\varphi}{\partial x_j}\left(\frac{x-w}{t}\right)dw\right)\cdot\psi(x,y,t)\,dx\frac{dt}{t^n}$$

with $\psi(x,y,t)$ bounded

But, using that $supp\varphi\subset B(0,\delta/4)$, we can see that the integration in t reduces to $t>c|x-y|$.

Using that

$$\int \frac{1}{t^{n+1}} \frac{\partial \varphi}{\partial x_j}\left(\frac{x-w}{t}\right) dw = 0$$

Using that

$$\int \frac{1}{t^{n+1}} \frac{\partial \varphi}{\partial x_j}\left(\frac{x-w}{t}\right) dw = 0$$

we can subtract $f(x)$ to obtain,

$$I \lesssim \int_{\mathbb{R}^n} \int_{c|x-y|}^{1} \left(\int_{|x-w|<\delta t/2} \frac{|f(w)-f(x)|}{t^{n+1}} \left| \nabla \varphi\left(\frac{x-w}{t}\right) \right| dw \right) \frac{dt}{t^n} dx$$

Using that

$$\int \frac{1}{t^{n+1}} \frac{\partial \varphi}{\partial x_j} \Big(\frac{x-w}{t} \Big) \, dw = 0$$

we can subtract $f(x)$ to obtain,

$$I \lesssim \int_{\mathbb{R}^n} \int_{c|x-y|}^{1} \left(\int_{|x-w|<\delta t/2} \frac{|f(w)-f(x)|}{t^{n+1}} \left| \nabla \varphi \Big(\frac{x-w}{t} \Big) \right| dw \right) \frac{dt}{t^n} dx$$

But $|x-w| \leq d(x)$ and so,

$$I$$

$$\lesssim \int_{\mathbb{R}^n} \int_{c|x-y|}^{1} \int_{|x-w|<d(x)} \frac{|f(w)-f(x)|}{|w-x|^{\frac{n}{p}+s}} \frac{1}{t^{n+\frac{n}{p'}+1-s}} \left| \nabla\varphi\left(\frac{x-w}{t}\right)\right| dwdtdx$$

$$\lesssim \int_{\frac{g(x)}{|x-y|^{n-s}}} dx$$

where

$$g(x) := \left(\int_{|x-w|<d(x)} \frac{|f(w)-f(x)|^p}{|w-x|^{n+ps}} dw \right)^{\frac{1}{p}}$$

$$I \lesssim \int_{\mathbb{R}^n} \int_{c|x-y|}^{1} \int_{|x-w|<d(x)} \frac{|f(w)-f(x)|}{|w-x|^{\frac{n}{p}+s}} \frac{1}{t^{n+\frac{n}{p'}+1-s}} \left|\nabla\varphi\left(\frac{x-w}{t}\right)\right| dwdtdx$$

$$\lesssim \int_{\frac{g(x)}{|x-y|^{n-s}}} dx$$

where

$$g(x) := \left(\int_{|x-w|<d(x)} \frac{|f(w)-f(x)|^p}{|w-x|^{n+ps}} dw\right)^{\frac{1}{p}}$$

Using this "representation formula" we can extend the argument given above to the fractional case.

MORE GENERAL WEIGHTS

The above arguments are based on continuity properties of the Hardy-Littlewood maximal function.

MORE GENERAL WEIGHTS

The above arguments are based on continuity properties of the Hardy-Littlewood maximal function.

With different arguments we proved (Acosta-Cejas-D.):

$$\|f - f_\Omega\|_{L^p_\omega(\Omega)} \leq C\|d\nabla f\|_{L^p_\omega(\Omega)}$$

for doubling weights ω which satisfies the local Poincaré

$$\|f - f_Q\|_{L^p_\omega(Q)} \leq C\|\nabla f\|_{L^p_\omega(Q)}$$

for whitney type cubes.

The above arguments are based on continuity properties of the Hardy-Littlewood maximal function.

With different arguments we proved (Acosta-Cejas-D.):

$$\|f - f_\Omega\|_{L^p_\omega(\Omega)} \le C\|d\nabla f\|_{L^p_\omega(\Omega)}$$

for doubling weights ω which satisfies the local Poincaré

$$\|f - f_Q\|_{L^p_\omega(Q)} \le C\|\nabla f\|_{L^p_\omega(Q)}$$

for whitney type cubes.

In particular the improved Poincaré is valid for the weights considered by FKS.

$$\begin{cases} -\mathrm{div}\,(\omega \nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\omega \nabla u \cdot n = f & \text{on } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}\,(\omega \nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\omega \nabla u \cdot n = f & \text{on } \Gamma_N \end{cases}$$

The mixed formulation is given by

$$\begin{cases} \boldsymbol{\sigma} + \omega \nabla u = 0 & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{\sigma} = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot n = f & \text{on } \Gamma_N \end{cases}$$

$$\begin{cases} -\mathrm{div}\,(\omega\nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\omega\nabla u\cdot n = f & \text{on } \Gamma_N \end{cases}$$

The mixed formulation is given by

$$\begin{cases} \boldsymbol{\sigma} + \omega\nabla u = 0 & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{\sigma} = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \boldsymbol{\sigma}\cdot n = f & \text{on } \Gamma_N \end{cases}$$

Notation:

$$H(\mathrm{div}\,,\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^n : \mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)\}$$

and,

$$H_{\Gamma_N}(\mathrm{div}\,,\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{div}\,,\Omega) : \ \boldsymbol{\tau}\cdot n = 0 \ \text{ on } \ \Gamma_N\}$$

Dividing by ω the first equation we obtain the mixed weak formulation:

Find $\boldsymbol{\sigma} \in H(\text{div}\,, \Omega)$ and $u \in L^2(\Omega)$ such that

Dividing by ω the first equation we obtain the mixed weak formulation:

Find $\boldsymbol{\sigma} \in H(\mathrm{div}\,,\Omega)$ and $u \in L^2(\Omega)$ such that

$$\boldsymbol{\sigma} \cdot n = f \quad \text{on } \Gamma_N$$

and

$$\begin{cases} \int_\Omega \omega^{-1}\,\boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,dx - \int_\Omega u\,\mathrm{div}\,\boldsymbol{\tau}\,dx = 0 & \forall \boldsymbol{\tau} \in H_{\Gamma_N}(\mathrm{div}\,,\Omega) \\ \int_\Omega v\,\mathrm{div}\,\boldsymbol{\sigma}\,dx = \int_\Omega gv\,dx & \forall v \in L^2(\Omega) \end{cases}$$

Recall that the Dirichlet boundary condition is implicit in the weak formulation (i. e., it is imposed in a natural way)

One of our motivations to analyze mixed approximations for degenerate problems was the fractional laplacian.

$$\begin{cases} (-\Delta)^s v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

THE FRACTIONAL LAPLACIAN

One of our motivations to analyze mixed approximations for degenerate problems was the fractional laplacian.

$$\begin{cases} (-\Delta)^s v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

This is a non local problem.
Caffarelli and Silvestre have shown that this problem is equivalent to a degenerate elliptic problem in $n+1$ variables: $v(x) = u(x,0)$ where If $u(x,y)$ is the solution of with $\alpha = 1-2s$

$$\begin{cases} \operatorname{div}(y^\alpha \nabla u(x,y)) = 0 & \text{in } D := \Omega \times (0,Y) \\ -\lim_{y\to 0} y^\alpha u_y = f & \text{on } \Gamma := \Omega \times \{0\} \\ u = 0 & \partial D \setminus \Gamma \end{cases}$$

In other words, the fractional laplacian is a Dirichlet to Neumann map.

In other words, the fractional laplacian is a Dirichlet to Neumann map.

Nochetto-Otárola-Salgado analyzed standard FEM approximation for these problems.

In other words, the fractional laplacian is a Dirichlet to Neumann map.

Nochetto-Otárola-Salgado analyzed standard FEM approximation for these problems.

Acosta-Borthagaray analyzed numerical approximations of the non-local problem.

In other words, the fractional laplacian is a Dirichlet to Neumann map.

Nochetto-Otárola-Salgado analyzed standard FEM approximation for these problems.

Acosta-Borthagaray analyzed numerical approximations of the non-local problem.

Our motivation to use mixed methods is that the variable $\sigma = y^\alpha \nabla u(x, y)$ seems to behave better than ∇u.

To illustrate the idea consider the trivial example

$$\left\{\begin{array}{rl} (y^\alpha u'(y))' = y^{-1/2} & \text{in } (0,1) \\ -\lim_{y\to 0} y^\alpha u'(y) = 1 & \\ u(1) = 0 & \end{array}\right.$$

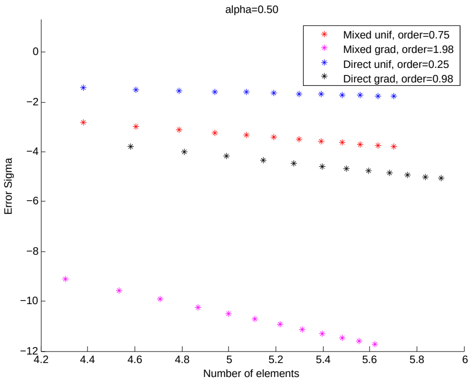
$$\sigma(y) = y^\alpha u'(y)$$

For example, taking $\alpha = 1/2$, we can see that the expected order for

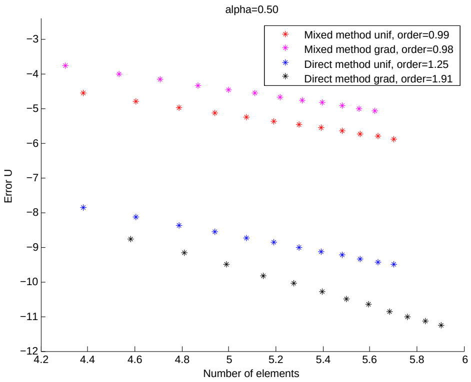
$$\int_0^1 (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)^2 y^{-\alpha}\, dy \sim \int_0^1 (u'(y) - u_h'(y))^2 y^\alpha\, dy$$

is $3/4$ the mixed method and $1/4$ for the standard one.

ERROR IN σ

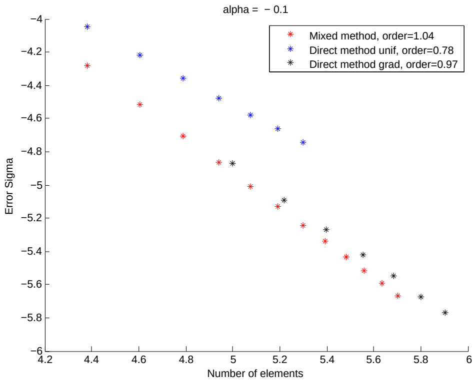


ERROR IN u



alpha=0.50
Mixed method unif, order=0.99
Mixed method grad, order=0.98
Direct method unif, order=1.25
Direct method grad, order=1.91
Error U
Number of elements
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
4.2
4.4
4.6
4.8
5
5.2
5.4
5.6
5.8
6

A NEGATIVE α



MIXED FEM APPROXIMATION

We will consider the approximation by the lowest order Raviart-Thomas space in n-dimensions.

We will consider the approximation by the lowest order Raviart-Thomas space in n-dimensions.

For a rectangular element R the local space is

$$RT_0(R) = \{\tau \in L^2(R)^n \, : \, \tau(x) = (a_1 + b_1 x_1, \cdots, a_n + b_n x_n)\}$$

MIXED FEM APPROXIMATION

We will consider the approximation by the lowest order Raviart-Thomas space in n-dimensions.

For a rectangular element R the local space is

$$RT_0(R) = \{\tau \in L^2(R)^n : \tau(x) = (a_1 + b_1 x_1, \cdots, a_n + b_n x_n)\}$$

and for a simplex T,

$$RT_0(T) = \{\tau \in L^2(T)^n : \tau(x) = (a_1 + b x_1, \cdots, a_n + b x_n)\}$$

MIXED FEM APPROXIMATION

We will consider the approximation by the lowest order Raviart-Thomas space in *n*-dimensions.

For a rectangular element R the local space is

$$RT_0(R) = \{\boldsymbol{\tau} \in L^2(R)^n \,:\, \boldsymbol{\tau}(x) = (a_1 + b_1 x_1, \cdots, a_n + b_n x_n)\}$$

and for a simplex T,

$$RT_0(T) = \{\boldsymbol{\tau} \in L^2(T)^n \,:\, \boldsymbol{\tau}(x) = (a_1 + b x_1, \cdots, a_n + b x_n)\}$$

Associated with a partition $\mathcal{T}_h$ of the domain Ω we introduce the global spaces

$$RT_0(\mathcal{T}_h) = \{\boldsymbol{\tau} \in H(\mathrm{div}\,,\Omega) : \boldsymbol{\tau}|_R \in RT_0(R) \;\; \forall R \in \mathcal{T}_h\}$$

and

$$\mathcal{P}_0(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_R \in \mathcal{P}_0(R) : \;\; \forall R \in \mathcal{T}_h\}$$

A fundamental tool for the error analysis is the well known Raviart-Thomas operator defined by

$$\int_F \Pi_h \boldsymbol{\tau} \cdot n \, dS = \int_F \boldsymbol{\tau} \cdot n \, dS$$

for all face F.

A fundamental tool for the error analysis is the well known Raviart-Thomas operator defined by

$$\int_F \Pi_h \boldsymbol{\tau} \cdot n \, dS = \int_F \boldsymbol{\tau} \cdot n \, dS$$

for all face F.
Π_h satisfies

$$\int_\Omega \operatorname{div}(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) \, v \, dx = 0 \qquad \forall v \in \mathcal{P}_0(\mathcal{T}_h)$$

Introducing the subspace

$$RT_{0,\Gamma_N}(\mathcal{T}_h) = RT_0(\mathcal{T}_h) \cap H_{\Gamma_N}(\mathrm{div}\,,\Omega),$$

and the orthogonal projection onto piecewise constants P_{Γ_N}, Introducing the subspace

$$RT_{0,\Gamma_N}(\mathcal{T}_h) = RT_0(\mathcal{T}_h) \cap H_{\Gamma_N}(\mathrm{div}\,,\Omega),$$

and the orthogonal projection onto piecewise constants P_{Γ_N}, the mixed finite element approximation is given by $(\boldsymbol{\sigma}_h, u_h) \in RT_0(\mathcal{T}_h) \times \mathcal{P}_0(\mathcal{T}_h)$ satisfying

$$\boldsymbol{\sigma}_h \cdot n = P_{\Gamma_N} f \qquad \text{on} \;\; \Gamma_N$$

and

$$\begin{cases} \int_\Omega \omega^{-1} \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}\, dx - \int_\Omega u_h \,\mathrm{div}\, \boldsymbol{\tau}\, dx = 0 & \forall \boldsymbol{\tau} \in RT_{0,\Gamma_N}(\mathcal{T}_h) \\ \int_\Omega v \,\mathrm{div}\, \boldsymbol{\sigma}_h\, dx = \int_\Omega g v\, dx & \forall v \in \mathcal{P}_0(\mathcal{T}_h) \end{cases}$$

ERROR ESTIMATES

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2_{\omega^{-1}}(\Omega)} \leq 2\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}(\Omega)}$$

and

$$\|u - u_h\|_{L^2_\omega(\Omega)} \leq C\left\{\|u - P_h u\|_{L^2_\omega(\Omega)} + \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}(\Omega)}\right\}$$

where P_h is the orthogonal L^2 projection onto $\mathcal{P}_0(\mathcal{T}_h)$.

ERROR ESTIMATES

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2_{\omega^{-1}}(\Omega)} \leq 2\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}(\Omega)}$$

and

$$\|u - u_h\|_{L^2_\omega(\Omega)} \leq C\left\{\|u - P_h u\|_{L^2_\omega(\Omega)} + \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}(\Omega)}\right\}$$

where P_h is the orthogonal L^2 projection onto $\mathcal{P}_0(\mathcal{T}_h)$.

The arguments are standard but we need the existence of $\boldsymbol{\tau} \in H^1_{\omega^{-1}}(\Omega)$ solution of

$$\operatorname{div}\boldsymbol{\tau} = (P_h u - u_h)\omega$$

satisfying

$$\|\boldsymbol{\tau}\|_{H^1_{\omega^{-1}}(\Omega)} \leq C\|(P_h u - u_h)\omega\|_{L^2_{\omega^{-1}}(\Omega)}$$

WHAT DO WE HAVE TO PROVE?

We need estimates for:

- $\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}}$

WHAT DO WE HAVE TO PROVE?

We need estimates for:

- $\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}}$
- $\|u - P_h u\|_{L^2_\omega(\Omega)}$

WHAT DO WE HAVE TO PROVE?

We need estimates for:

- $\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}}$
- $\|u - P_h u\|_{L^2_\omega(\Omega)}$

And we need a continuous right inverse of

$$\text{div} \; : \; H^1_{0,\omega^{-1}} \to L^2_{\omega^{-1}}$$

We need estimates for:

- $\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L^2_{\omega^{-1}}}$
- $\|u - P_h u\|_{L^2_\omega(\Omega)}$

And we need a continuous right inverse of

$$\mathrm{div} \ : H^1_{0,\omega^{-1}} \to L^2_{\omega^{-1}}$$

Moreover, we want anisotropic error estimates (for example for the application to the fractional Laplacian).

As we have seen, error estimates follows from Poincaré type inequalities.

As we have seen, error estimates follows from Poincaré type inequalities.

What can be said in the anisotropic case?

As we have seen, error estimates follows from Poincaré type inequalities.

What can be said in the anisotropic case?

Strong A_2 condition or A_2^s:

$$[\omega]_{A_2^s} := \sup_R \left(\frac{1}{|R|} \int_R \omega \, dx \right) \left(\frac{1}{|R|} \int_R \omega^{-1} \, dx \right) < \infty$$

where the sup is taken over all rectangles with sides parallel to the coordinate axes.

ANISOTROPIC WEIGHTED ESTIMATES

Consider an arbitrary rectangle

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \qquad h_i = b_i - a_i$$

$d_i(x) := \min\{(b_i - x_i), (x_i - a_i)\}$.

ANISOTROPIC WEIGHTED ESTIMATES

Consider an arbitrary rectangle

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \qquad h_i = b_i - a_i$$

$d_i(x) := \min\{(b_i - x_i), (x_i - a_i)\}.$

It is known that the constant in the improved Poincaré inequality for R blows up when the ratio between outer and inner diameter goes to infinity.

Consider an arbitrary rectangle

$$R = [a_1, b_1] \times \cdots \times [a_n, b_n] \qquad h_i = b_i - a_i$$

$d_i(x) := \min\{(b_i - x_i), (x_i - a_i)\}$.

It is known that the constant in the improved Poincaré inequality for R blows up when the ratio between outer and inner diameter goes to infinity.

However, we have the following anisotropic version if the weight belongs to the smaller class A_2^s.

For $\omega \in A_2^s$,

$$\|v - v_R\|_{L^2_\omega(R)} \leq C_\omega \sum_{i=1}^{n} \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(R)}$$

For $\omega \in A_2^s$,

$$\|v - v_R\|_{L^2_\omega(R)} \leq C_\omega \sum_{i=1}^{n} \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(R)}$$

Indeed, it follows immediately from the improved Poincaré inequality that, if Q is the unitary cube,

$$\|v - v_Q\|_{L^2_\omega(Q)} \leq C_\omega \sum_{i=1}^{n} \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(Q)}$$

For $\omega \in A_2^s$,

$$\|v - v_R\|_{L^2_\omega(R)} \leq C_\omega \sum_{i=1}^n \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(R)}$$

Indeed, it follows immediately from the improved Poincaré inequality that, if Q is the unitary cube,

$$\|v - v_Q\|_{L^2_\omega(Q)} \leq C_\omega \sum_{i=1}^n \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(Q)}$$

Then, the above anisotropic version for R follows by standard arguments making the change of variables $x_i = h_i \hat{x}_i$ and using that, for $\hat{\omega}(\hat{x}) := \omega(x)$, $\omega \in A_2^s \Longrightarrow \hat{\omega} \in A_2^s$.

GENERALIZED WEIGHTED POINCARÉ INEQUALITIES

For $\omega \in A_2^s$ and F the face contained in $x_1 = a_1$ we have

$$\|v - v_F\|_{L^2_\omega(R)} \leq C_\omega \left\{ \left\| (b_1 - x_1) \frac{\partial v}{\partial x_1} \right\|_{L^2_\omega(R)} + \sum_{i=2}^{n} \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(R)} \right\}$$

GENERALIZED WEIGHTED POINCARÉ INEQUALITIES

For $\omega \in A_2^s$ and F the face contained in $x_1 = a_1$ we have

$$\|v - v_F\|_{L^2_\omega(R)} \le C_\omega \left\{ \left\|(b_1 - x_1)\frac{\partial v}{\partial x_1}\right\|_{L^2_\omega(R)} + \sum_{i=2}^{n} \left\| d_i \frac{\partial v}{\partial x_i}\right\|_{L^2_\omega(R)} \right\}$$

Proof:

By a simple integration by parts we have

$$\frac{1}{|F|}\int_F v\, dS = \frac{1}{|R|}\int_R v\, dx + \frac{1}{|R|}\int_R (x_1 - b_1)\frac{\partial v}{\partial x_1}\, dx$$

GENERALIZED WEIGHTED POINCARÉ INEQUALITIES

Then,

$$v - v_F = v - v_R - \frac{1}{|R|}\int_R (x_1 - b_1)\frac{\partial v}{\partial x_1}\,dx$$

and therefore,

$$\|v - v_F\|_{L^2_\omega(R)} \le \|v - v_R\|_{L^2_\omega(R)} + \int_R (b_1 - x_1)\left|\frac{\partial v}{\partial x_1}\right|\,dx\,\frac{1}{|R|}\left(\int_R \omega\,dx\right)^{1/2}$$

but, multiplying and dividing by $\omega^{1/2}$ and using the Schwarz inequality we obtain

$$\int_R (b_1 - x_1)\left|\frac{\partial v}{\partial x_1}\right|\,dx \le \left\|(b_1 - x_1)\frac{\partial v}{\partial x_1}\right\|_{L^2_\omega(R)}\left(\int_R \omega^{-1}\,dx\right)^{1/2}$$

then,

$$\|v - v_F\|_{L^2_\omega(R)} \leq \|v - v_R\|_{L^2_\omega(R)} + [\omega]^{1/2}_{A^s_2} \left\| (b_1 - x_1) \frac{\partial v}{\partial x_1} \right\|_{L^2_\omega(R)}$$

GENERALIZED WEIGHTED POINCARÉ INEQUALITIES

then,

$$\|v - v_F\|_{L^2_\omega(R)} \leq \|v - v_R\|_{L^2_\omega(R)} + [\omega]_{A_2^s}^{1/2} \left\| (b_1 - x_1) \frac{\partial v}{\partial x_1} \right\|_{L^2_\omega(R)}$$

and therefore

$$\|v - v_F\|_{L^2_\omega(R)} \leq C_\omega \left\{ \left\| (b_1 - x_1) \frac{\partial v}{\partial x_1} \right\|_{L^2_\omega(R)} + \sum_{i=2}^{n} \left\| d_i \frac{\partial v}{\partial x_i} \right\|_{L^2_\omega(R)} \right\}$$

Since $\boldsymbol{\sigma}_j - \Pi\boldsymbol{\sigma}_j$ has vanishing mean value on the face defined by $x_j = a_j$ we obtain the following error estimate for the Raviart-Thomas interpolation of lowest order:

Since $\boldsymbol{\sigma}_j - \Pi\boldsymbol{\sigma}_j$ has vanishing mean value on the face defined by $x_j = a_j$ we obtain the following error estimate for the Raviart-Thomas interpolation of lowest order:

For $\omega \in A_2^s$ and $1 \leq j \leq n$,

$$\|\boldsymbol{\sigma}_j - \Pi_h\boldsymbol{\sigma}_j\|_{L^2_\omega(R)} \leq C_\omega \sum_{i=1}^{n} \left\|(b_i - a_i)\frac{\partial \boldsymbol{\sigma}_j}{\partial x_i}\right\|_{L^2_\omega(R)}$$

Question: which ω are in A_2^s?

Question: which ω are in A_2^s?

$\omega \in A_2^s$ iff w belongs to A_2 of one variable for each variable uniformly in the other variables (Kurtz).

Question: which ω are in A_2^s?

$\omega \in A_2^s$ iff w belongs to A_2 of one variable for each variable uniformly in the other variables (Kurtz).

Observe that this is the case for the weight y^α, $-1 < \alpha < 1$ appearing in the fractional laplacian.

Question: which ω are in A_2^s?

$\omega \in A_2^s$ iff w belongs to A_2 of one variable for each variable uniformly in the other variables (Kurtz).

Observe that this is the case for the weight y^α, $-1 < \alpha < 1$ appearing in the fractional laplacian.

Or more generally,

$$\omega(x) = \omega_1(x_1) \cdots \omega_n(x_n)$$

with

$$\omega_i(x_i) \in A_2(\mathbb{R})$$

To finish the error analysis we need also to show the existence of a solution of

$$\operatorname{div} \boldsymbol{\tau} = v$$

satisfying

$$\|\boldsymbol{\tau}\|_{H^1_\omega(\Omega)} \leq C\|v\|_{L^2_\omega(\Omega)}$$

To finish the error analysis we need also to show the existence of a solution of

$$\operatorname{div} \boldsymbol{\tau} = v$$

satisfying

$$\|\boldsymbol{\tau}\|_{H^1_\omega(\Omega)} \leq C\|v\|_{L^2_\omega(\Omega)}$$

But, the same representation formula given above define the Bogovski solution:

$$\boldsymbol{\tau}(x) = \int_\Omega G(x,y)v(y)\,dy$$

To finish the error analysis we need also to show the existence of a solution of

$$\operatorname{div} \boldsymbol{\tau} = v$$

satisfying

$$\|\boldsymbol{\tau}\|_{H^1_\omega(\Omega)} \leq C\|v\|_{L^2_\omega(\Omega)}$$

But, the same representation formula given above define the Bogovski solution:

$$\boldsymbol{\tau}(x) = \int_\Omega G(x, y)v(y)\,dy$$

The estimate for $\|\boldsymbol{\tau}\|_{H^1_\omega(\Omega)}$ can be proved using the continuity of Calderón-Zygmund integral operators in weighted norms for A_2 weights.

In conclusion we obtain the forllowing error estimates for the RT_0 approximation of

$$\begin{cases} -\text{div}\,(\omega \nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\omega \nabla u \cdot n = f & \text{on } \Gamma_N \end{cases}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L^2_{\omega^{-1}}(\Omega)} \leq C_\omega \sum_{R \in \mathcal{T}_h} \sum_{i=1}^{n} \left\| (b_i - x_i) \frac{\partial \boldsymbol{\sigma}}{\partial x_i} \right\|^2_{L^2_{\omega^{-1}}(R)}$$

or

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L^2_{\omega^{-1}}(\Omega)} \leq C_\omega \sum_{R \in \mathcal{T}_h} \sum_{i=1}^{n} h_i^2 \left\| \frac{\partial \boldsymbol{\sigma}}{\partial x_i} \right\|^2_{L^2_{\omega^{-1}}(R)}$$

For this case we can prove that, if Ω is convex, for $i, j = 1, \cdots, n$,

$$\frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial y}, \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial x_j}, \frac{\partial \boldsymbol{\sigma}_i}{\partial x_j} \in L^2_{y^{-\alpha}}$$

while, for $i = 1, \cdots, n$

$$\frac{\partial \boldsymbol{\sigma}_i}{\partial y} \in L^2_{y^{-\alpha+\beta}}$$

for $\beta > 1 - \alpha$.
Therefore, we can obtain optimal order convergence using a graded mesh.

We use

$$\int |D\boldsymbol{\sigma}|^2 y^{-\alpha+\beta} \leq C$$

for $\beta > 1 - \alpha$.

For the elements in the first band $R = R_x \times [0, y_1]$ we use

$$\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|^2_{L^2_{y^{-\alpha}}(R)} \leq h_1^{2-\beta} \int |D\boldsymbol{\sigma}|^2 y^{-\alpha+\beta}$$

and we choose $h_1 = h^{\frac{2}{2-\beta}}$.

For the rest of the elements $R = R_x \times [y_j, y_{j+1}]$ we choose $y_{j+1} = y_j + h y_j^{\gamma}$

$$\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|^2_{L^2_{y^{-\alpha}}(R)} \leq (y_{j+1} - y_j)^2 \int |D\boldsymbol{\sigma}|^2 y^{-\alpha}\, dy$$

$$\leq h^2 y_j^{2\gamma} \int |D\boldsymbol{\sigma}|^2 y^{-\alpha}\, dy \leq h^2 \int |D\boldsymbol{\sigma}|^2 y^{-\alpha+2\gamma}\, dy$$

We have to choose $\gamma = \beta/2$. But we need also $\gamma < 1$. Then, we have to take $\frac{1-\alpha}{2} < \gamma < 1$

For the rest of the elements $R = R_x \times [y_j, y_{j+1}]$ we choose $y_{j+1} = y_j + hy_j^\gamma$

$$\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L^2_{y^{-\alpha}}(R)} \leq (y_{j+1} - y_j)^2 \int |D\boldsymbol{\sigma}|^2 y^{-\alpha}\, dy$$

$$\leq h^2 y_j^{2\gamma} \int |D\boldsymbol{\sigma}|^2 y^{-\alpha}\, dy \leq h^2 \int |D\boldsymbol{\sigma}|^2 y^{-\alpha+2\gamma}\, dy$$

We have to choose $\gamma = \beta/2$. But we need also $\gamma < 1$. Then, we have to take
$\frac{1-\alpha}{2} < \gamma < 1$

Remark: $h \sim 1/N$ where N is the number of nodes in the y direction, i.e., the error estimate is optimal with respect to the number of nodes.

MUCHAS GRACIAS !